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Robustness of entanglement for Bell decomposable states

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Abstract. We propose a simple geometrical approach for finding robustness of entanglement for Bell decomposable states of two-qubit quantum systems. It is shown that for these states robustness is equal to the concurrence. We also present an analytical expression for two separable states that wipe out all entanglement of these states. Random robustness of these states is also obtained. We also obtain robustness of a class of states obtained from Bell decomposable states *via* some special local operations and classical communications (LOCC).

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.)

1 Introduction

During past decade an increasing study has been made on entanglement, although it was discovered several decades ago by Einstein and Schrödinger [1,2]. This is because of the central role that entanglement plays in the theory of quantum information [3–5]. Entanglement as the most non classical features of quantum mechanics is usually arise from quantum correlations between separated subsystems which can not be created by local actions on each subsystem. By definition, a mixed state ρ of a bipartite system is said to be separable (non entangled) if it can be written as a convex combination of pure product states

$$\rho = \sum_i p_i \left|\phi_i^A\right\rangle \left\langle\phi_i^A\right| \otimes \left|\psi_i^B\right\rangle \left\langle\psi_i^B\right|, \qquad (1)$$

where $\left|\phi_i^A\right\rangle$ and $\left|\psi_i^B\right\rangle$ are pure states of subsystems A and B, respectively. Although, in the case of pure states of bipartite systems it is easy to check whether a given state is, or is not entangled, the question is yet an open problem in the case of mixed states.

There is also an increasing attention in quantifying entanglement, particularly for mixed states of a bipartite system, and a number of measures have been proposed [5–8]. Among them the entanglement of formation has more importance, since it intends to quantify the resources needed to create a given entangled state.

One useful quantity introduced in [9] as a measure of entanglement is robustness of entanglement. It corresponds to the minimal amount of mixing with separable states which washes out all entanglement. Analytical expression for pure states of binary systems have given in [9]. Authors in [10] gave a geometrical interpretation of robustness and pointed that two corresponding separable states needed to wipe out all entanglement are necessarily on the boundary of separable set. Unfortunately, above mentioned quantity as most proposed measures of entanglement involves extremization which are difficult to handle analytically.

In this paper we consider Bell decomposable (BD) states. We provide a simple geometrical approach and give an analytic expression for robustness of entanglement and show that the corresponding separable states are on the boundary of separable states as pointed out in [10]. Our approach to the calculation of robustness of entanglement is geometrically intuitive. It is shown that for considered states robustness is equal to the concurrence of states. We also obtain random robustness for BD states. By defining new norm based on spin flip operation and using some local operations and classical communications (LOCC), we also obtain robustness of some particular classes of mixed states.

The paper is organized as follows. In Section 2 we review BD states and present a perspective of their geometry. Robustness of entanglement of these states is obtained in Section 3 *via* a geometrical approach. We also obtain random robustness. Definition of new norm and robustness of some special two-qubit states, obtained *via* LOCC, is presented in Section 4. The paper is ended with a brief conclusion in Section 5.

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2 Bell decomposable states

In this section we briefly review Bell decomposable (BD) states and some of their properties (for a detail see Ref. [11]). A BD state is defined by:

$$\rho = \sum_{i=1}^{4} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \quad 0 \le p_i \le 1, \quad \sum_{i=1}^{4} p_i = 1, \tag{2}$$

where $|\psi_i\rangle$ is Bell state, given by:

$$|\psi_1\rangle = \left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \tag{3}$$

$$|\psi_2\rangle = \left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle), \tag{4}$$

$$|\psi_3\rangle = \left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \tag{5}$$

$$|\psi_4\rangle = \left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{6}$$

In Hilbert-Schmidt representation ρ can be written as

$$\rho = \frac{1}{4}\left(I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i\right), \tag{7}$$

where

$$\begin{aligned} t_1 &= p_1 - p_2 + p_3 - p_4, \\ t_2 &= -p_1 + p_2 + p_3 - p_4, \\ t_3 &= p_1 + p_2 - p_3 - p_4. \end{aligned} \tag{8}$$

From positivity of ρ we get

$$\begin{aligned} 1 + t_1 - t_2 + t_3 &\ge 0, \\ 1 - t_1 + t_2 + t_3 &\ge 0, \\ 1 + t_1 + t_2 - t_3 &\ge 0, \\ 1 - t_1 - t_2 - t_3 &\ge 0. \end{aligned} \tag{9}$$

These equations form a tetrahedral with its vertices located at $(1,-1,1)$, $(-1,1,1)$, $(1,1,-1)$, $(-1,-1,-1)$ [11]. In fact these vertices correspond to the Bell states of equations (3–6), respectively.

According to the Peres and Horodecki's condition for separability [12,13], a two-qubit density matrix is separable if and only if its partial transpose is positive. This implies that ρ given in equation (7) is separable if and only if t_i satisfy equations (9) together with the following equations

$$\begin{aligned} 1 + t_1 + t_2 + t_3 &\ge 0, \\ 1 - t_1 - t_2 + t_3 &\ge 0, \\ 1 + t_1 - t_2 - t_3 &\ge 0, \\ 1 - t_1 + t_2 - t_3 &\ge 0. \end{aligned} \tag{10}$$

As Horodeckis have shown in reference [11], inequalities (9, 10) form an octahedral with its vertices located at $O_1^{\pm} = (\pm 1, 0, 0)$, $O_2^{\pm} = (0, \pm 1, 0)$ and $O_3^{\pm} = (0, 0, \pm 1)$.

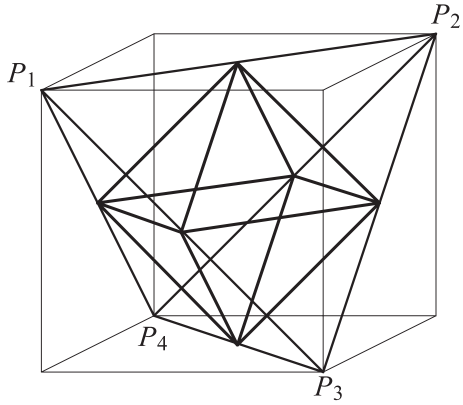

Fig. 1. All BD states are defined as points interior to tetrahedral. Vertices P_1, P_2, P_3 and P_4 denote projectors corresponding to Bell states equations (3–6), respectively. Octahedral corresponds to separable states.

This means that tetrahedral of equations (9) is divided into five regions. Central regions, defined by octahedral, are separable states. There are also four smaller equivalent tetrahedral corresponding to entangled states. Each tetrahedral takes one Bell state as one of its vertices. Three other vertices of each tetrahedral form a triangle which is its common face with octahedral (see Fig. 1).

As far as entanglement is concerned the states ρ and ρ' are equivalent if they are on the same orbit of the group of local unitary transformation, that is, if there exist local unitary transformation $U_1 \otimes U_2$ such that $\rho' = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^{\dagger}$, where U_1 and U_2 are unitary transformations acting on Hilbert spaces of particles A and B, respectively. It is easy to show that for any unitary transformation U there is a unique rotation O such that [11]

$$U\, \hat{\mathbf{n}} \cdot \sigma\, U^{\dagger} = (O\, \hat{\mathbf{n}}) \cdot \sigma. \tag{11}$$

Now if the state given in equation (7) is subjected to the $U_1 \otimes U_2$ transformations the matrix $T = \mathrm{diag}(t_1, t_2, t_3)$ is no longer diagonal and transforms as

$$T' = O_1 T O_2^T, \tag{12}$$

where O_i denote rotation matrices correspond to the local unitary matrices U_i. This means that all the results which we are going to obtain for Bell decomposable states given in equation (7) also satisfy for arbitrary density matrix with T' give in equation (12).

3 Robustness of entanglement

According to [9] for a given entangled state ρ and separable state ρ_s, a new density matrix $\rho(s)$ can be constructed as,

$$\rho(s) = \frac{1}{s+1}(\rho + s\rho_s), \quad s \ge 0, \tag{13}$$

where it can be either entangled or separable. It was pointed that there always exits the minimal s corresponding to ρ_s such that $\rho(s)$ is separable. This minimal s

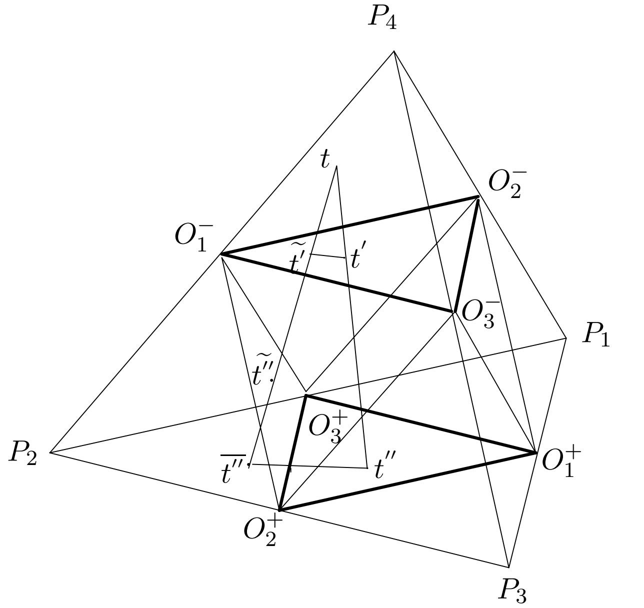

Fig. 2. Two local pseudomixture for entangled state ρ (denoted with point t). Relative robustness of ρ with respect to states lying on the separable plan $O_1^+O_2^+O_3^+$ are minimum.

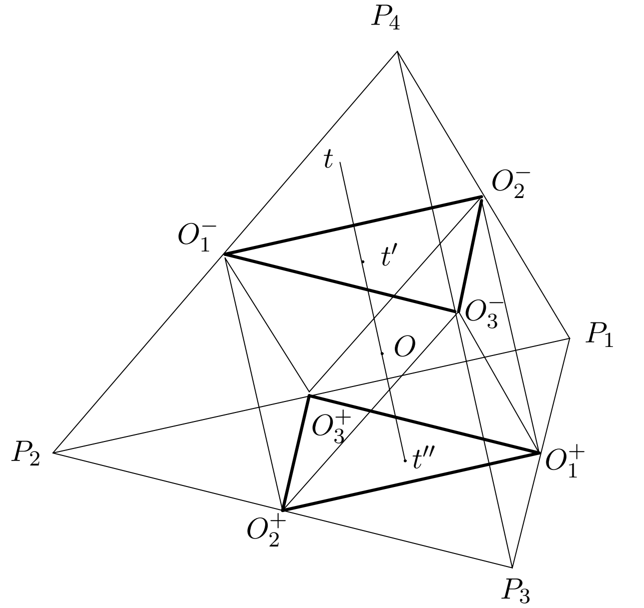

Fig. 3. Point t denotes a generic state ρ in entangled region dominated by singlet state $|\psi^-\rangle$ and points t' and t'' are, respectively, on the separable boundary planes defined by equations $x_1 + x_2 + x_3 + 1 = 0$ and $x_1 + x_2 + x_3 - 1 = 0$.

is called the robustness of ρ relative to ρ_s, denoted by $R(\rho \parallel \rho_s)$. The absolute robustness of ρ is defined as the quantity,

$$R(\rho \parallel S) \equiv \min_{\rho_s \in S} R(\rho \parallel \rho_s). \tag{14}$$

Du *et al.* in [10] gave a geometrical interpretation of robustness and pointed that if s in equation (13) is minimal among all separable states ρ_s, *i.e.* s is the absolute robustness of ρ, then ρ_s and $\rho(s)$ in equation (13) are necessarily on the boundary of the separable states.

In this section we obtain absolute robustness for all Bell diagonal states, and give an explicit form for the corresponding ρ_s and $\rho(s)$ which are on the boundary of the separable states. To this aim, we first suppose that separable state ρ_s that minimize equation (14) lies on octahedral. Consider point t, corresponding to density matrix ρ, in entangled region dominated by singlet state $|\psi^-\rangle$ (see Fig. 2). We draw two lines from the point t, one that cuts separable plane $O_1^+O_2^+O_3^+$ at t'' and the other that cuts separable plane $O_1^-O_2^+O_3^+$ at $\widetilde{t''}$, where its extension cuts plane $P_1P_2P_3$ at $\overline{t''}$. These lines cut separable plane $O_1^-O_2^-O_3^-$ at points t' and $\widetilde{t'}$, respectively. Since planes $O_1^-O_2^-O_3^-$ and $O_1^+O_2^+O_3^+$ are parallel, it follows that

$$\frac{|\, tt' \,|}{|\, t't'' \,|} = \frac{|\, t\tilde{t}' \,|}{|\, \tilde{t}'\overline{t}'' \,|} \leq \frac{|\, t\tilde{t}' \,|}{|\, \tilde{t}'\tilde{t}'' \,|}, \tag{15}$$

that is, robustness of ρ relative to separable states lying on the plane $O_1^+O_2^+O_3^+$ is less than those which lie on the plane $O_1^-O_2^+O_3^+$. Similarly we can obtain the same result for other separable planes $O_1^+O_2^+O_3^-$ and $O_1^+O_2^-O_3^+$. This means that as far as robustness of ρ relative to separable states ρ_s defined by octahedral is concerned, those separable states that lie on the plane $O_1^+O_2^+O_3^+$ have minimum robustness. Before we show that this is, indeed, the minimum over all possible separable states, we try to calculate it. We first note that as above argument show, robustness of ρ relative to all separable states that lie on the plane $O_1^+O_2^+O_3^+$ are equal. To give a general framework for ρ_s we note that plane $O_1^+O_2^+O_3^+$ is reflection of plane $O_1^-O_2^-O_3^-$ with respect to octahedral center. Considering this fact, we connect t, corresponding to density matrix ρ, to the center of octahedral such that cuts the plane $O_1^-O_2^-O_3^-$ at t' (see Fig. 3). Then we extend this segment, where it cuts the other plane $O_1^+O_2^+O_3^+$ at t''.

Three points t, t' and t'' are along the same line but they posses different lengths. Also it is not difficult to see that they also lie on planes $x_1+x_2+x_3+\eta = 0$, $x_1+x_2+x_3+1 = 0$ and $x_1+x_2+x_3-1$, respectively. Using the above argument, we arrive after some elementary algebra at the following results

$$t_i' = \frac{t_i}{\eta} = \frac{-t_i}{t_1+t_2+t_3}, \tag{16}$$

$$t_i'' = \frac{-t_i}{\eta} = \frac{t_i}{t_1+t_2+t_3}. \tag{17}$$

Now using the convexity of the set of density matrices, we can write ρ_s' as,

$$\rho_s' = \frac{1}{1+s}(\rho + s\rho_s''), \tag{18}$$

where parameter s, called robustness of ρ, can be obtained as

$$s = \frac{|\, t\, t' \,|}{|\, t'\, t'' \,|} = \frac{-(1+t_1+t_2+t_3)}{2} = 2p_4 - 1 = C, \tag{19}$$

where equations (8) have been used and C is concurrence of ρ [8]. Also as we mentioned above robustness of ρ relative to all separable states lying in the separable plane $O_1^+O_2^+O_3^+$ is equal to that given in equation (19). Among them there exist some states ρ that can be simultaneously written as

$$\rho = (1+s)\rho'_s - s\rho''_s = (1-\lambda)\left|\psi^-\right\rangle\left\langle\psi^-\right| + \lambda\rho''_s \tag{20}$$

where $(1-\lambda) = s = C$ (for detail see Ref. [14]). For the second decomposition, called Lewenstein-Sanpera decomposition, authors in [15] have shown that average concurrence of the decomposition is equal to the concurrence for all two-qubit density matrices.

Vidal *et al.* in [9] have shown that $R(\rho \parallel \rho_s)$ is a convex function of ρ_s. This means that any local minimum is also the absolute one, thus to finding absolute minimum of $R(\rho \parallel \rho_s)$ as a function of ρ_s it is enough to find local minimum.

As we have already shown the robustness given in equation (19) is minimum over all Bell decomposable states. Here we have to show that it is minimum with respect to all separable states. Let us consider the generic separable states ρ'_s and ρ''_s defined by

$$\rho'_s = \sum_i p'_i \left|\psi_i\right\rangle\left\langle\psi_i\right| + \sum_{i,j} a_{ij}\left|\psi_i\right\rangle\left\langle\psi_j\right| \tag{21}$$

$$\rho''_s = \sum_i p''_i \left|\psi_i\right\rangle\left\langle\psi_i\right| + \sum_{i,j} b_{ij}\left|\psi_i\right\rangle\left\langle\psi_j\right| \tag{22}$$

where $a_{ii} = b_{ii} = 0$ for $i = 1,2,3,4$. Now in order to writing ρ as pseudomixture of the above separable states

$$\rho = (1+s)\rho'_s - s\rho''_s, \tag{23}$$

the following equations must hold

$$p_i = (1+s)p'_i - sp''_i, \tag{24}$$

$$(1+s)a_{ij} - s\,b_{ij} = 0. \tag{25}$$

Now one can easily obtain robustness of ρ relative to ρ'_s as

$$s = \frac{\parallel \rho - \rho'_s \parallel}{\parallel \rho'_s - \rho''_s \parallel} = \sqrt{\frac{\sum_i (p_i - p'_i)^2 + \mathrm{Tr}(AA^\dagger)}{\sum_i (p'_i - p''_i)^2 + \mathrm{Tr}(A-B)(A-B)^\dagger}} = \sqrt{\frac{\sum_i (p_i - p'_i)^2 + \mathrm{Tr}(AA^\dagger)}{\sum_i (p'_i - p''_i)^2 + \frac{1}{s^2}\mathrm{Tr}(AA^\dagger)}}, \tag{26}$$

where A and B are matrices with matrix elements a_{ij} and b_{ij}, respectively, and in the last line we have used equation (25). By solving equation (26) for robustness **s** we get

$$s = \sqrt{\frac{\sum_i (p_i - p'_i)^2}{\sum_i (p'_i - p''_i)^2}}. \tag{27}$$

Equation (27) shows that off-diagonal elements of ρ'_s and ρ''_s (in basis that ρ is diagonal) play no role in robustness. This means that the robustness given in equation (19) is local minimum, thus according to reference [9] it is absolute minimum.

In the pioneering paper [9], robustness of entanglement for Werner states as a special kind of BD states (that is, $p_1 = p_2 = p_3 = (1-p_4)/3$, where $p_4 = F$ is their fidelity), has been obtained from an entirely different approach. We see that the treatment applied for Werner states leads to the same answer as obtained in [9].

Finally, We would like to emphasis that our treatment is capable to give explicit expression for separable matrices ρ'_s and ρ''_s. Since, using equations (16, 17, 7) we can write ρ'_s and ρ''_s as

$$\rho'_s = \frac{1}{4(t_1+t_2+t_3)} \times \begin{pmatrix} t_1+t_2 & 0 & 0 & -t_1+t_2 \\ 0 & t_1+t_2+2t_3 & -t_1-t_2 & 0 \\ 0 & -t_1-t_2 & t_1+t_2+2t_3 & 0 \\ -t_1+t_2 & 0 & 0 & t_1+t_2 \end{pmatrix}, \tag{28}$$

$$\rho''_s = \frac{1}{4(t_1+t_2+t_3)} \times \begin{pmatrix} t_1+t_2+2t_3 & 0 & 0 & t_1-t_2 \\ 0 & t_1+t_2 & t_1+t_2 & 0 \\ 0 & t_1+t_2 & t_1+t_2 & 0 \\ t_1-t_2 & 0 & 0 & t_1+t_2+2t_3 \end{pmatrix}. \tag{29}$$

Also Vidal and Tarrach [9] have defined another quantity called random robustness, which is defined as robustness of ρ relative to maximally random state I/n. For Bell decomposable states considered here we can evaluate it as follows. Using the convexity of the set of density matrices, we can write ρ'_s as (see Fig. 3),

$$\rho'_s = \frac{1}{1+s_0}(\rho + s_0\,\rho_0), \tag{30}$$

where $\rho_0 = I/4$ and

$$s_0 = \frac{|\,t\,t'\,|}{|\,t'\,O\,|} = -(1+t_1+t_2+t_3) = 2(2p_4-1) = 2\,C, \tag{31}$$

is random robustness of ρ. Note that for the states considered here separable matrix ρ'_s is same as the one given in equation (28) but with $\rho''_s = I/4$.

It follows from equation (30) that arbitrary density matrix $\rho_\epsilon = \epsilon\rho + (1-\epsilon)\rho_0$ is separable provided that ϵ is sufficiently small *i.e.*, $\epsilon \leq 1/(1+s_0)$. This means that random robustness of all entangled states can be used to obtain upper bound for the size of neighborhood of the maximally random density matrix. The problem that there exists a sufficiently small neighborhood of the maximally random density matrix inside which all density matrices are separable is addressed in reference [16]. In reference [17] upper and lower bounds on the size of neighborhood and implications of the bounds for NMR quantum computing has been shown.

4 Robustness of entanglement under LOCC

In this section we obtain robustness for a class of states which can be obtain from BD states *via* some local operations and classical communications (LOCC). Our approach is based on the fact that a full rank two qubit density matrix can be obtained from a unique Bell decomposable state by using a suitable local filtering operation on one single copy. This idea has been already used in reference [18] to parameterize the manifold of states with constant concurrence in order to obtain the lowest possible value of negativity for a given concurrence. Also, the relation between local filtering operation and Bell violation has been shown in reference [19].

A general LOCC operation is defined by multi-local super operator that does not increase the trace. Mathematically, a general LOCC operation can be represented on a bipartite state ρ by [20,21]

$$\rho' = \frac{\sum_i (A_i \otimes B_i)\rho(A_i \otimes B_i)^\dagger}{\mathrm{Tr}\left(\sum_i (A_i \otimes B_i)\rho(A_i \otimes B_i)^\dagger\right)}, \tag{32}$$

where $\sum_i A_i^\dagger A_i \leq 1$ and $\sum_i B_i^\dagger B_i \leq 1$. In this paper we restrict ourself to the case that LOCC operation is represented by single local filtering, *i.e.*

$$\rho' = \frac{(A \otimes B)\rho(A \otimes B)^\dagger}{\mathrm{Tr}((A \otimes B)\rho(A \otimes B)^\dagger)}, \tag{33}$$

where operators A and B can be written as

$$A \otimes B = U_A\, f^{\mu,a,\mathbf{m}} \otimes U_B\, f^{\nu,b,\mathbf{n}}, \tag{34}$$

where U_A and U_B are unitary operators acting on subsystems A and B, respectively and the filtration f is defined by

$$\begin{aligned} f^{\mu,a,\mathbf{m}} &= \mu(I_2 + a\,\mathbf{m}\cdot\sigma), \\ f^{\nu,b,\mathbf{n}} &= \nu(I_2 + b\,\mathbf{n}\cdot\sigma). \end{aligned} \tag{35}$$

We now perform LOCC of the form (33) on BD states given in equation (2) to obtain transformed density matrix

$$\overline{\rho} = \sum_i \overline{p}_i \left|\overline{\psi}_i\right\rangle \left\langle\overline{\psi}_i\right|, \tag{36}$$

where unnormalized vectors $\left|\overline{\psi}_i\right\rangle$ are defined by

$$\left|\overline{\psi}_i\right\rangle = (A \otimes B)\left|\psi_i\right\rangle, \tag{37}$$

such that

$$\left\langle\overline{\psi}_i|\overline{\psi}_j\right\rangle = K_{ij}, \tag{38}$$

and

$$\overline{p}_i = \frac{p_i}{\mathrm{Tr}((A \otimes B)\rho(A \otimes B)^\dagger)},$$
$$\mathrm{Tr}\left((A \otimes B)\rho(A \otimes B)^\dagger\right) = \sum_i p_i K_{ii}. \tag{39}$$

As defined by Wootters in [8] states $\left|\widetilde{\psi}\right\rangle$ can be written as

$$\left|\widetilde{\psi}\right\rangle = (\sigma_y \otimes \sigma_y)\left|\psi^*\right\rangle, \tag{40}$$

where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ when it is expressed in a standard basis such as $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}, \{|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and σ_y represent Pauli matrix in local basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Using equations (37, 40), we can easily seen that

$$\begin{aligned} \left\langle\overline{\psi}_i|\widetilde{\overline{\psi}}_j\right\rangle &= \det(A)\det(B)\left\langle\psi_i|\widetilde{\psi}_i\right\rangle \\ &= \det(A)\det(B)\eta_i\delta_{ij}, \end{aligned} \tag{41}$$

where we have used $A^\dagger\sigma_y A^* = \det(A)\sigma_y$, and $\eta_2 = \eta_3 = -\eta_1 = -\eta_4 = 1$. As it is shown in references [20,21], the concurrence of the state ρ transforms under LOCC of the form given in equation (33) as

$$C(\overline{\rho}) = \frac{\det(A)\det(B)}{\mathrm{Tr}((A \otimes B)\rho(A \otimes B)^\dagger)}\, C(\rho). \tag{42}$$

Now consider the following pseudomixture for $\overline{\rho}$

$$\overline{\rho} = (1+\overline{s})\rho_s' - \overline{s}\rho_s'', \tag{43}$$

where

$$\rho_s' = \sum_i p_i' \left|\overline{\psi}_i\right\rangle\left\langle\overline{\psi}_i\right| + \sum_{i,j} M_{ij}\left|\overline{\psi}_i\right\rangle\left\langle\overline{\psi}_j\right|, \tag{44}$$

$$\rho_s'' = \sum_i p_i'' \left|\overline{\psi}_i\right\rangle\left\langle\overline{\psi}_i\right| + \sum_{i,j} N_{ij}\left|\overline{\psi}_i\right\rangle\left\langle\overline{\psi}_j\right|. \tag{45}$$

From equation (43) it follows that following equations should be hold

$$\overline{p}_i = (1+\overline{s})p_i' - \overline{s}p_i'', \tag{46}$$

$$(1+\overline{s})M_{ij} - \overline{s}N_{ij} = 0. \tag{47}$$

In the sequel we want to obtain robustness using a new norm defined by

$$\|A\| := \sqrt{\mathrm{Tr}(A\,\tilde{A})}, \tag{48}$$

where $\tilde{A}$ is defined according to equation (40). With respect to this norm the distance between two density matrices ρ_1 and ρ_2 is defined by

$$\|\rho_1 - \rho_2\| = \sqrt{\mathrm{Tr}((\rho_1 - \rho_2)(\tilde{\rho}_1 - \tilde{\rho}_2))}. \tag{49}$$

By using equation (49) robustness of $\overline{\rho}$ relative to ρ''_s is defined by

$$\overline{s} = \frac{\| \overline{\rho} - \rho'_s \|}{\| \rho'_s - \rho''_s \|} = \sqrt{\frac{\mathrm{Tr}(\overline{\rho} - \rho'_s)(\widetilde{\overline{\rho}} - \widetilde{\rho'_s})}{\mathrm{Tr}(\rho'_s - \rho''_s)(\widetilde{\rho'_s} - \widetilde{\rho''_s})}}. \quad (50)$$

Using equations (44, 45, 47) we get

$$\overline{s} = \sqrt{\frac{\sum_i (\overline{p}_i - p'_i)^2 + \mathrm{Tr}(MM^*)}{\sum_i (p'_i - p''_i)^2 + \mathrm{Tr}(M-N)(M-N)^*}}$$

$$= \sqrt{\frac{\sum_i (\overline{p}_i - p'_i)^2 + \mathrm{Tr}(MM^*)}{\sum_i (p'_i - p''_i)^2 + \frac{1}{\overline{s}^2}\mathrm{Tr}(MM)^*}}, \quad (51)$$

where it can be solved for $\overline{s}$

$$\overline{s} = \sqrt{\frac{\sum_i (\overline{p}_i - p'_i)^2}{\sum_i (p'_i - p''_i)^2}}. \quad (52)$$

This equation shows that in LOCC transformed density matrix $\overline{\rho}$ given in equation (36) only diagonal elements of ρ'_s and ρ''_s play role in robustness when it is evaluated with the norm defined by (48). We now restrict ourselves to special LOCC operations for which $K_{ii} = K$ for $i = 1, 2, 3, 4$. To this end we use representation given in equation (7) for Bell states and we get

$$K_{ii} = \mathrm{Tr}(\langle \overline{\psi}_i | \overline{\psi}_i \rangle)$$
$$= \mu^2 \nu^2 \left((1+a^2)(1+b^2) + 4ab \sum_j t_j^{(i)} m_j n_j \right), \quad (53)$$

where $t_j^{(i)}$ denote coordinates of Bell state $|\psi_i\rangle$ in tetrahedral, namely $(1,-1,1)$, $(-1,1,1)$, $(1,1,-1)$ and $(-1,-1,-1)$, respectively. After some algebra we can see that conditions $K_{ii} = K$ happen in the cases that

$$m_i n_i = 0 \ \text{ for } \ i = 1, 2, 3, \quad \text{or} \quad ab = 0. \quad (54)$$

Under this restriction we get

$$\overline{p}_i = \frac{p_i}{K}. \quad (55)$$

On the other hand normalization conditions of ρ'_s and ρ''_s lead to

$$\sum_i p'_i = \frac{1}{K}, \qquad \sum_i p''_i = \frac{1}{K}. \quad (56)$$

Equations (55, 56) show that in order to obtain robustness of $\overline{\rho}$ it is enough to restrict ourselves to a new tetrahedral where its dimensions is contracted by factor $1/K$. Using the similar procedure that we did for Bell decomposable states, we can easily evaluate robustness for these states

$$\overline{s} = 2p_4 - 1 = \frac{K}{\det(A)\det(B)} C(\overline{\rho}), \quad (57)$$

i.e. under restricted LOCC operation robustness does not change.

5 Conclusion

In this work we have obtained robustness of entanglement for Bell decomposable states. It is shown that for these states robustness is equal to their concurrence. It is also shown that the corresponding separable states that wipe out all entanglement of the states are on the boundary of separable states. The random robustness of these states is also obtained. By defining a new norm based on spin flip operation, robustness of a class of states obtained *via* some LOCC operations is also obtained. The LOCC operation which we consider in this paper is restricted to single filtering satisfying equation (54). The general local filtering and its effect on robustness of entanglement has been investigated in [22]. Also the effect of a general LOCC operation with multi-local super operator on robustness of entanglement is yet an open problem which is under investigations.

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